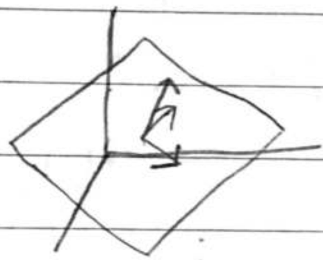
lecture 4 §12.4 - cross product

- our goal: to answer a geometric question



• given 2 vectors: (in R^3)

$\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$

construct a vector

$\vec{w} = \langle w_1, w_2, w_3 \rangle$ such that

$\vec{w}$ is orthogonal to $\vec{u}$ and $\vec{v}$

- <u>How?</u> we know $\begin{cases} ① \; 0 = \vec{u} \cdot \vec{w} = u_1 w_1 + u_2 w_2 + u_3 w_3 \\ ② \; 0 = \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \end{cases}$

lets eliminate w_3

① mutiply eq① by v_3 and eq② by u_3 to obtain:

$$\begin{cases} 0 = v_3(\vec{u} \cdot \vec{w}) = (u_1 v_3) w_1 + (u_2 v_3) w_2 + (u_3 v_3) w_3 \\ 0 = u_3(\vec{v} \cdot \vec{w}) = (u_3 v_1) w_1 + (u_3 v_2) w_2 + (u_3 v_3) w_3 \end{cases}$$

② subtract eq② from eq① to get:

$$0 = v_3(\vec{u} \cdot \vec{w}) - u_3(\vec{v} \cdot \vec{w})$$

$$= (u_1 v_3 - u_3 v_1) w_1 + (u_2 v_3 - u_3 v_2) w_2$$

eq③: $-(-(u_1 v_3 - u_3 v_1)) w_1 + (u_2 v_3 - u_3 v_2) w_2$

$$-ax + by = 0 \quad \text{if } x = b \text{ and } y = a$$

$$-ab + ba = 0$$

hence eq (3) has a solution

$$\begin{cases} w_1 = u_2v_3 - u_3v_2 \\ w_2 = -(u_1v_3 - u_3v_1) \end{cases}$$

· inputting these into eq (1), we obtain:

$$0 = u_1w_1 + u_2w_2 + u_3w_3$$

$$0 = u_1(u_2v_3 - u_3v_2) + u_2(-(u_1v_3 - u_3v_1)) + u_3v_3$$

$$0 = \cancel{u_1u_2v_3} - u_1u_3v_2 - \cancel{u_1u_2v_3} + u_2u_3v_1 + u_3v_3$$

$$u_3(u_2v_1 - u_1v_2 + u_3v_3$$

Hence either $u_3 = 0$ or $w_3 = u_1v_2 - u_2v_1$

- Claim: (ignoring the fact that u_3 may be 0) we have a solution:

$$\vec{w} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle$$

the check is left to us.

- Def (Determinant): the determinant of the 2x2 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ matrix is $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

• The determinant of a 3x3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det \begin{bmatrix} -1 & 3 & 7 \\ 0 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$-1 \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 7 \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

$$-1(-1(1) + 1(0)) - 3(0 \cdot 1 - -1 \cdot 1) + 7(0 \cdot 1 - (-1) \cdot 1)$$

$$-1(-1+0) - 3(0-1) + 7(+1)$$

$$1 \quad + 3 \quad 7 = 11$$

- <u>cross product definition:</u>
  - let $\vec{u} = \langle u_1, v_2\ u_3 \rangle$, $\vec{v} = \langle v_1 v_2\ v_3 \rangle \in R^3$
  - cross product of $\vec{u}$ with v is

$$\vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \vec{i} \begin{vmatrix} v_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \vec{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \vec{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$(u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

$$= \langle u_2 v_3 - u_3 v_2) - \langle u_1 v_3 - u_3 v_1 \rangle + \langle u_1 v_2 - u_2 v_1 \rangle$$

- This has all been done in R^3.....
  - This cross product only works in R^3....
  - The cross product is a vector operation.

(vector <u>in R^3</u> x vector <u>in R^3</u> $\longmapsto$ vector <u>in R^3</u>

- Algebraic properties of x product :

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}^3$

① $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

② $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

③ $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

④ $(\vec{u} + \vec{v}) \times w = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

⑤ $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

⑥ $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

- Geometric properties of cross product

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$

① $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

② $|\vec{u} \times \vec{v}|$ - magnitude - is $|\vec{u}||\vec{v}| \sin(\theta)$ where θ is the angle between them.

③ $\vec{u} \times \vec{v} = \vec{0}$ iff $\vec{u}$ is parallel to $\vec{v}$.